The Renormalization-group Method Applied to Non-equilibrium Dynamics

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Abstract

We review the renormalization group method applied to non-equilibrium dynamics by tracing the way how the hydrodynamic equations can be derived as reduced dynamics of the Boltzmann equation as a typical example.

1 Introduction

The theory of the nonequilibrium dynamics may be regarded as a collection of the theory how to reduce the dynamics of many-body systems to ones with fewer variables [1]. In fact, the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy[2] can be reduced to the time-*ir*reversible Boltzmann equation[3], which is given solely in terms of the single-particle distribution function for dilute gas systems[4]. The derivation of Boltzmann equation by Bogoliubov[4] shows that the dilute-gas dynamics as a *dynamical system* with many-degrees of freedom has an *attractive manifold* [5] spanned by the one-particle distribution function, which is also an *invariant manifold* [5]. Boltzmann equation in turn can be further reduced to the hydrodynamic equation (Euler or Navier-Stokes equation[6]) by a perturbation theory like Chapman-Enskog method[7] or Bogoliubov's method[4, 8]. Langevin equation[9] which may be time-irreversible can be reduced to the time-irreversible Fokker-Planck equation with a longer time scale than the scale in Langevin equation[9].

Two basic ingredients are commonly seen in the reduction of dynamics, which are interrelated with but relatively independent of each other: (i) The reduced dynamics is characterized with a longer time scale than that appearing in the original (microscopic) evolution equation. (ii) The reduced dynamics is described by a time-irreversible equation even when the original microscopic equation is time-reversible[10, 11]. The fundamental problem in the theory of deriving kinetic or transport equations is to clarify the mechanism of and to implement the above two basic ingredients.

In a few years ago, we showed[12] that the so called the renormalization-group method[13, 14, 15], which is formulated in terms of the classical theory of envelopes[16], gives a unified theory for the reduction of the dynamics and can be used to derive the various transport equations in a transparent way; we have also elucidated that the underlying mathematics of the reduction is an explicit construction of the invariant manifold[12, 17], a notion in dynamical system[5, 8].

In this report, we pick up the problem to derive the hydrodynamics equations from the Boltzmann equation as a typical problem of the reduction appearing in the field of non-equilibrium dynamics. Some remarks will be also given on applications of the method to other problems such as the reduction of the dynamics of Fokker-Planck equation and also on an extension to the relativistic case.

2 Derivation of hydrodynamic equations from Boltzmann equation

In this section, we apply the RG method formulated in [16, 17] to derive Euler and Navier-Stokes equations, successively from the Boltzmann equation[12].

2.1 Basics of Boltzmann equation

Boltzmann equation[3, 18] is an evolution equation of the one-particle distribution function $f(\boldsymbol{r}, \boldsymbol{v}, t)$ in the phase space, and reads

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} = I[f]. \tag{1}$$

Here the left-hand side describes the change due to the canonical equation of motion while the right-hand side the change due to collisions;

$$\begin{aligned} I[f] &= \int d\boldsymbol{v}_1 \int d\boldsymbol{v}' \int d\boldsymbol{v}'_1 w(\boldsymbol{v}\,\boldsymbol{v}_1|\boldsymbol{v}'\,\boldsymbol{v}'_1) \\ &\quad \times \Big\{ f(\boldsymbol{r}, \boldsymbol{v}', t) f(\boldsymbol{r}, \boldsymbol{v}'_1, t) \\ &- f(\boldsymbol{r}, \boldsymbol{v}, t) f(\boldsymbol{r}, \boldsymbol{v}_1, t) \Big\}, \end{aligned} \tag{2}$$

which is called the collision integral. The transition probability $w(\boldsymbol{v}\,\boldsymbol{v}_1|\boldsymbol{v}'\,\boldsymbol{v}'_1)$ has the following symmetry due to the time-reversal invariance of the microscopic equation of motion; $w(\boldsymbol{v}\,\boldsymbol{v}_1|\boldsymbol{v}'\,\boldsymbol{v}'_1) = w(\boldsymbol{v}'\,\boldsymbol{v}'_1|\boldsymbol{v}\,\boldsymbol{v}_1)$. Furthermore, the invariance under the particle-interchange implies the following equality; $w(\boldsymbol{v}\,\boldsymbol{v}_1|\boldsymbol{v}'\,\boldsymbol{v}'_1) = w(\boldsymbol{v}_1\,\boldsymbol{v}|\boldsymbol{v}'_1\,\boldsymbol{v}') = w(\boldsymbol{v}'_1\,\boldsymbol{v}'|\boldsymbol{v}_1\,\boldsymbol{v})$.

We say that the function $\varphi(\boldsymbol{v})$ is a *collision invariant* if it satisfies the following equation;

$$\int d\boldsymbol{v} \varphi(\boldsymbol{v}) I[f] = 0. \tag{3}$$

Owing to the conservation of the particle number, the total momentum and the total kinetic energy during the collisions, the quantities given by linear combination of the following five quantities are collision invariant: $1, \boldsymbol{v}$ and v^2.

For a collision invariant $\varphi(\boldsymbol{v})$, we can define the density n_φ and the current $\boldsymbol{j}_\varphi$ as follows;

$$\begin{aligned} n_\varphi &= \int d\boldsymbol{v} \varphi(\boldsymbol{v}) f(\boldsymbol{r}, \boldsymbol{v}, t), \\ \boldsymbol{j}_\varphi &= \int d\boldsymbol{v} \boldsymbol{v} \varphi(\boldsymbol{v}) f(\boldsymbol{r}, \boldsymbol{v}, t), \end{aligned} \tag{4}$$

which satisfy the continuity or balance equation;

$$\partial_t n_\varphi + \nabla \cdot \boldsymbol{j}_\varphi = 0. \tag{5}$$

Thus we have formally the hydrodynamic equations as the balance equations for the conservation of the particle number, total momentum and kinetic energy. These equations are, however, formal ones because the distribution function f is not yet solved: The solution obtained from Boltzmann equation will give the explicit forms of the internal energy, the transport coefficients and so on.

The H function is defined as follows:

$$H(\boldsymbol{r}, t) = \int d\boldsymbol{v}\, f(\boldsymbol{r}, \boldsymbol{v}, t)(\ln f(\boldsymbol{r}, \boldsymbol{v}, t) - 1). \tag{6}$$

For equilibrium states, the H function is equal to the entropy S with the sign changed. Defining the corresponding current by

$$\boldsymbol{J}_H(\boldsymbol{r}, t) = \int d\boldsymbol{v}\, \boldsymbol{v}\, f(\boldsymbol{r}, \boldsymbol{v}, t)(\ln f(\boldsymbol{r}, \boldsymbol{v}, t) - 1), \tag{7}$$

one has the balance equation;

$$\frac{\partial H}{\partial t} + \nabla \cdot \boldsymbol{J}_H = \int d\boldsymbol{v}\, I[f] \ln f. \tag{8}$$

This shows that when $\ln f$ is a collision invariant, the H function is conserved. In fact, one can show that this condition is satisfied when $f(\boldsymbol{r}, \boldsymbol{v}, t)$ is a local equilibrium distribution function as given by (19) below.

2.2 Application of the RG method to derive hydrodynamic equations

To make it clear that the following discussion fits to the general formulation given in [17], we discretize the argument $\boldsymbol{v}$ as $\boldsymbol{v} \to \boldsymbol{v}_i$[1]: Discriminating the arguments $(\boldsymbol{r}, t)$ and $\boldsymbol{v}_i$ in $f(\boldsymbol{r}, \boldsymbol{v}_i, t)$, we indicate $\boldsymbol{v}_i$ as a subscript i for the distribution function; $f(\boldsymbol{r}, \boldsymbol{v}_i, t) = f_i(\boldsymbol{r}, t) \equiv (\boldsymbol{f}(\boldsymbol{r}, t))_i$. Then Boltzmann equation now reads

$$\frac{\partial f_i}{\partial t} = \hat{I}[\boldsymbol{f}]_i - \boldsymbol{v}_i \cdot \frac{\partial f_i}{\partial \boldsymbol{r}}, \tag{9}$$

where

$$\hat{I}[\boldsymbol{f}]_i = \sum_{j,k,l} w(\boldsymbol{v}_i \boldsymbol{v}_j | \boldsymbol{v}_k \boldsymbol{v}_l)(f_k f_l - f_i f_j)(\boldsymbol{r}, t). \tag{10}$$

Now let us consider a situation where the fluid motion is slow with long wave-lengths so that

$$\boldsymbol{v}_i \cdot \frac{\partial f_i}{\partial \boldsymbol{r}} = O(\epsilon), \tag{11}$$

where ϵ is a small quantity,$|\epsilon| < 1$. To take into account the smallness of ϵ in the following calculations formally, let us introduce the scaled coordinate $\bar{\boldsymbol{r}}$ defined by $\bar{\boldsymbol{r}} = \epsilon \boldsymbol{r}$ and $\partial/\partial \boldsymbol{r} = \epsilon \cdot \partial/\partial \bar{\boldsymbol{r}}$. Then (9) reads

$$\frac{\partial f_i}{\partial t} = \hat{I}[\boldsymbol{f}]_i - \epsilon \boldsymbol{v}_i \cdot \frac{\partial f_i}{\partial \bar{\boldsymbol{r}}}, \tag{12}$$

which has a form to which the perturbation theory given in [17] is naturally applicable.

In accordance with the general formulation given in [17], we first expand the solution as follows; $f_i(\bar{\boldsymbol{r}}, t) = f_i^{(0)}(\bar{\boldsymbol{r}}, t) + \epsilon f_i^{(1)}(\bar{\boldsymbol{r}}, t) + \cdots$. Let

$\tilde{f}_i(\bar{\boldsymbol{r}},t;t_0))$ be a solution around $t\sim t_0$ given by a perturbation theory with $f_i(\bar{\boldsymbol{r}},t_0)$ being the initial value at $t=t_0$;

$$\tilde{f}_i(\bar{\boldsymbol{r}},t=t_0;t_0)=f_i(\bar{\boldsymbol{r}},t_0). \tag{13}$$

We expand $\tilde{f}_i(\bar{\boldsymbol{r}},t;t_0)$ as

$$\begin{aligned}\tilde{f}_i(\bar{\boldsymbol{r}},t;t_0) &= \tilde{f}_i^{(0)}(\bar{\boldsymbol{r}},t;t_0)+\epsilon\tilde{f}_i^{(1)}(\bar{\boldsymbol{r}},t,t_0)\\ &+ \cdots,\end{aligned} \tag{14}$$

and the respective initial condition is set up as follows,

$$\tilde{f}_i^{(l)}(\bar{\boldsymbol{r}},t=t_0;t_0)=f_i^{(l)}(\bar{\boldsymbol{r}},t_0),\quad (l=0,1,2...). \tag{15}$$

The 0-th order equation reads

$$\frac{\partial\tilde{f}_i^{(0)}}{\partial t}=(I[\tilde{\boldsymbol{f}}^{(0)}])_i. \tag{16}$$

Now we are interested in the slow motion which may be achieved asymptotically as $t\to\infty$. Therefore we take the following stationary solution,

$$\frac{\partial\tilde{f}_i^{(0)}}{\partial t}=0, \tag{17}$$

which is a **fixed point** of the equation satisfying

$$(\hat{I}[\tilde{\boldsymbol{f}}^{(0)}])_i=0, \tag{18}$$

for arbitrary $\bar{\boldsymbol{r}}$. Notice that (18) shows that the distribution function $\tilde{\boldsymbol{f}}^{(0)}$ is a function of collision invariants. Such a distribution function is a local equilibrium distribution function or Maxwellian;

$$\begin{aligned}\tilde{f}_i^{(0)}(\bar{\boldsymbol{r}},t;t_0) &= n(\bar{\boldsymbol{r}},t_0)\Big(\frac{m}{2\pi k_BT(\bar{\boldsymbol{r}},t_0)}\Big)^{3/2}\\ &\times \exp\Big[-\frac{m|\boldsymbol{v}_i-\mathbf{u}(\bar{\boldsymbol{r}},t_0)|^2}{2\pi k_BT(\bar{\boldsymbol{r}},t_0)}\Big]\end{aligned} \tag{19}$$

Here, the local density n, local temperature T, local flux $\mathbf{u}$ are all dependent on the initial time t_0 and the space coordinate $\bar{\boldsymbol{r}}$ but independent of time t.

The first-order equation reads

$$\Big((\frac{\partial}{\partial t}-A)\tilde{f}^{(1)}\Big)_i=-\boldsymbol{v}_i\cdot\frac{\partial\tilde{f}_i^{(0)}}{\partial\bar{\boldsymbol{r}}}. \tag{20}$$

Here the linear operator A is defined by

$$\begin{aligned}\Big[\hat{I}'[\tilde{\boldsymbol{f}}^{(0)}]\tilde{\boldsymbol{f}}^{(1)}\Big]_i &= \sum_{j=1}^{\infty}\frac{\partial I}{\partial\tilde{f}_j}\Big|_{\tilde{\boldsymbol{f}}=\tilde{\boldsymbol{f}}^0}\cdot\tilde{f}_j^{(1)}\\ &\equiv (A\tilde{\boldsymbol{f}}^{(1)})_i.\end{aligned} \tag{21}$$

Defining the inner product between $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ by

$$\langle\boldsymbol{\varphi},\boldsymbol{\psi}\rangle=\int d\boldsymbol{v}\varphi\psi, \tag{22}$$

one can show that A is self-adjoint;

$$\langle\varphi,A\psi\rangle=\langle A\varphi,\psi\rangle. \tag{23}$$

One can further show that the five invariants $m,\boldsymbol{v},\frac{m}{2}v^2$ span the kernel of A[18]; KerA $=\{m,\boldsymbol{v},\frac{m}{2}\boldsymbol{v}^2\}$. The other eigenvalues are found to be negative because one can show

$$\langle\varphi,A\varphi\rangle \quad\le\quad 0. \tag{24}$$

We write the projection operator to the kernel as P and define Q $=1-$ P.

Applying the general formulation in [17], one can readily obtain the first-order solution,

$$\tilde{\boldsymbol{f}}^{(1)}=-(t-t_0)\mathrm{P}\boldsymbol{v}\cdot\frac{\partial\tilde{\boldsymbol{f}}^{(0)}}{\partial\bar{\boldsymbol{r}}}+A^{-1}\mathrm{Q}\boldsymbol{v}\cdot\frac{\partial\tilde{\boldsymbol{f}}^{(0)}}{\partial\bar{\boldsymbol{r}}}. \tag{25}$$

The perturbative solution up to the ϵ order is found to be

$$\begin{aligned}\tilde{\boldsymbol{f}}(\bar{\boldsymbol{r}},t,t_0) &= \tilde{\boldsymbol{f}}^{(0)}(\bar{\boldsymbol{r}},t,t_0)+\epsilon[-(t-t_0)\mathrm{P}\boldsymbol{v}\cdot\frac{\partial\tilde{\boldsymbol{f}}^{(0)}}{\partial\bar{\boldsymbol{r}}}\\ &+A^{-1}\mathrm{Q}\boldsymbol{v}\cdot\frac{\partial\tilde{\boldsymbol{f}}^{(0)}}{\partial\bar{\boldsymbol{r}}}].\end{aligned} \tag{26}$$

Notice the appearance of a secular term.

If one stops the approximation and apply the RG equation[16],

$$\partial\tilde{\boldsymbol{f}}/\partial t_0|_{t_0=t}=\mathbf{0}, \tag{27}$$

one has

$$\frac{\partial\tilde{\boldsymbol{f}}^{(0)}}{\partial t}+\epsilon\mathrm{P}\boldsymbol{v}\cdot\frac{\partial\tilde{\boldsymbol{f}}^{(0)}}{\partial\bar{\boldsymbol{r}}}=\mathbf{0}. \tag{28}$$

This is a master equation from which equations governing the time evolution of $n(\boldsymbol{r},t)$, $\mathbf{u}(\boldsymbol{r},t)$ and $T(\boldsymbol{r},t)$ in $\tilde{\boldsymbol{f}}^{(0)}$, i.e., fluid dynamical equations: In fact, taking an inner product between m, $m\boldsymbol{v}$ and $mv^2/2$ with this equation, one has fluid dynamical equations as given below. Notice that f is now explicitly solved and the energy density e, the pressure tensor P_{ij} and the heat flux Q_i are given as follows;

$$\begin{aligned}e(\boldsymbol{r},t) &= \int d\boldsymbol{v}\frac{m}{2}|\boldsymbol{v}-\mathbf{u}|^2f^{(0)}(\boldsymbol{r},\boldsymbol{v},t)\\ &= \frac{3}{2}k_BT(\boldsymbol{r},t),\\ P_{ij}(\boldsymbol{r},t) &= \int d\boldsymbol{v}m(v_i-u_i)(v_j-u_j)f^{(0)}(\boldsymbol{r},\boldsymbol{v},t)\end{aligned} \tag{29}$$

$$
\begin{aligned}
&= nk_BT(\boldsymbol{r},t)\delta_{ij} \\
&\equiv P(\boldsymbol{r},t)\delta_{ij}, \qquad (30)\\
Q_i(\boldsymbol{r},t) &= \int d\boldsymbol{v}\frac{m}{2}|\boldsymbol{v}-\mathbf{u}|^2(v_i-u_i)f^{(0)}(\boldsymbol{r},\boldsymbol{v},t)\\
&= 0. \qquad (31)
\end{aligned}
$$

We have defined the pressure P using the equation of state for the ideal gas in the second line. There is no heat flux because the distribution function $f^{(0)}$ in the formulae is the one for the local equilibrium. Inserting these formulae into (5), we end up with a fluid dynamical equation without dissipation, i.e., the **Euler equation**;

$$
\begin{aligned}
\frac{\partial\rho}{\partial t} &+ \nabla\cdot\rho\mathbf{u}=0, \qquad (32)\\
\frac{\partial(\rho u_i)}{\partial t} &+ \frac{\partial}{\partial x_j}(\rho u_iu_j)+\frac{\partial}{\partial x_i}P=0, \qquad (33)\\
\frac{\partial}{\partial t}(\rho u^2+e) &+ \frac{\partial}{\partial x_i}\left[(\rho\frac{u^2}{2}+e+P)u_i\right]=0 \qquad (34)
\end{aligned}
$$

We notice that these equations have been obtained from the RG equation (28). It should be emphasized however that the distribution function obtained in the present approximation takes the form

$$
\begin{aligned}
f(\boldsymbol{r},\boldsymbol{v},t) &= f^{(0)}(\boldsymbol{r},\boldsymbol{v},t)\\
&+ A^{-1}\mathrm{Q}\boldsymbol{v}\cdot\frac{\partial f^{(0)}(\boldsymbol{r},\boldsymbol{v},t)}{\partial\boldsymbol{r}}, \qquad (35)
\end{aligned}
$$

which incorporates as a perturbation a distortion from the local equilibrium distribution and gives rise to dissipations.

One can proceed to the second order approximation straightforwardly and obtain fluid dynamical equation with dissipations as the RG equation. The perturbation equation in the second order reads

$$
\left((\frac{\partial}{\partial t}-A)\tilde{f}^{(2)}\right)_i=-\boldsymbol{v}_i\cdot\frac{\partial\tilde{f}_i^{(1)}}{\partial\bar{\boldsymbol{r}}}. \qquad (36)
$$

Here, we must make an important notice: We have actually used the linearized Boltzmann equation[18] neglecting the second-order term of $\tilde{\boldsymbol{f}}^{(1)}$ in the collision integral: It is known that the neglected term produces the so called Burnett terms which are absent in the usual Navier-Stokes equation [6].

Applying the RG method, we have

$$
\begin{aligned}
\frac{\partial f^{(0)}}{\partial t} &+ \epsilon\mathrm{P}\boldsymbol{v}\cdot\nabla f^{(0)}\\
&+\epsilon^2\mathrm{P}\boldsymbol{v}\cdot\nabla A^{-1}\mathbf{Q}\boldsymbol{v}\cdot\nabla f^{(0)}=0. \qquad (37)
\end{aligned}
$$

The third term represents dissipations. Taking inner products between the first equation and the particle number, the velocity and the kinetic energy, one will obtain a fluid dynamical equation with dissipations included, i.e., Navier-Stokes equation[18].

3 Summary and concluding remarks

In summary, we have shown that the fluid dynamical limit of the Boltzmann equation can be obtained neatly in the RG method as formulated in [17]. It would be intriguing to see what kind of the dissipative fluid dynamical would emerge when the RG method is applied to the relativistic kinetic equations[19].

The reduction of a kinetic equation to a further slower dynamics appear quite often, reflecting the hierarchy of the space-time of nature. A typical problem [9, 20] in this category is to derive Smoluchowski equation[21] from Kramers equation[22]. As is expected, it is possible to develop a systematic theory for the adiabatic elimination of fast variables in Fokker-Planck equations, which appear in the theory of the critical dynamics[23].

We hope that we can report a development on these problems in near future.

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